Math 2050, 14 Sep

Using the same trick as before, we can define $m^{\frac{1}{n}}$ for any $m, n \in \mathbb{N}$ (or more generally the rational power of positive real number):

Example 0.1. *There exists $u \in \mathbb{R}$ such that $u > 0$ and $u^3 = 4$.*

Ans. Let $S = \{a \in \mathbb{R} : a^3 < 4, a > 0\}$. Clearly, S is non-empty as $1 \in S$. S is bounded from above since if $a \in S$, then $a \leq 2$ otherwise $a \geq 2$ and hence $a^3 \geq 8 > 4$ which is impossible. So by completeness, there is $v = \sup S \in \mathbb{R}$. Moreover, the same argument shows that $0 < 1 \leq v \leq 2$.

We claim that $v^3 = 4$. If $v^3 > 4$, then $u = v - \varepsilon$ satisfies

$$\begin{aligned} u^3 &= v^3 - 3\varepsilon v^2 + 3\varepsilon^2 v - \varepsilon^3 \\ &> v^3 - 12\varepsilon - \varepsilon^3. \end{aligned} \tag{0.1}$$

Therefore, if we choose ε to be a real number such that $0 < \varepsilon < \min\{1, \frac{v^3-4}{13}\}$, then $u^3 > 4$. Hence, u is an upper bound of S by the same reasoning as above which is impossible.

If $v^3 < 4$, then let $u = v + \varepsilon$ so that

$$\begin{aligned} u^3 &= v^3 + 3\varepsilon v^2 + 3\varepsilon^2 v + \varepsilon^3 \\ &\leq v^3 + 12\varepsilon + 6\varepsilon^2 + \varepsilon^3. \end{aligned} \tag{0.2}$$

Then if $\varepsilon > 0$ is chosen to be smaller than $\min\{1, \frac{4-v^3}{20}\}$, then $u^3 < 4$ and hence $u \in S$ which is impossible.

Therefore, $v^3 = 4$ which is what we want.

□

Proposition 0.1. *If S is non-empty subset which is bounded from below, then*

(1) *For any $a \in \mathbb{R}$, $\sup(a + S)$ exists and equal to $a + \sup(S)$;*
(2) *For any $a > 0$, $\sup(aS)$ exists and equal to $a \cdot \sup S$.*

Ans. (1): Clearly, $a + S$ is non-empty and bounded from above so that $\sup(a + S)$ exists in $\mathbb{R}$. By definition, for any $s \in S$, $a + s \in a + S$ and hence

$$a + s \leq \sup(a + S).$$

Therefore, $s \leq \sup(a + S) - a$ for all $s \in S$. Thus, $\sup(S) + a \leq \sup(a + S)$. Similarly, for any $s \in S$, $\sup(S) \geq s$ and hence

$$a + s \leq \sup(S) + a.$$

Therefore, $\sup(a+S) \leq \sup(S)+a$. Combined two inequalities, we are done.

(2): If $a > 0$. For any $s \in S$, $s \leq \sup(S)$ and hence

$$as \leq a \cdot \sup(S).$$

Therefore, $\sup(aS) \leq a \cdot \sup(S)$. Similarly, for any $as \in aS$, $as \leq \sup(aS)$ and thus, $s \leq \frac{1}{a}\sup(aS)$. Therefore, $\sup(S) \leq \frac{1}{a}\sup(aS)$. Combines two inequalities, we are done. $\square$

Question 0.1. *How to find $\sqrt{2}$ numerically?*

Our usual procedure: trial and error using rational number!
Trial 1. $1^2 = 1 < 2$. ($a_1 = 1$)
Trial 2. $1.2^2 = 1.44 < 2$ ($a_2 = 1.2$)
Trial 3. $1.3^2 = 1.69 < 2(a_3 = 1.3)$
Trial 4. $1.4^2 = 1.96 < 2(a_4 = 1.4)$
Trial 5. $1.41^2 = 1.9881 < 2(a_5 = 1.41)$
Trial n. etc....

This suggests approximation scheme of any real number using rational number.

Theorem 0.1 (Density of rational number)**.** *For any $x < y$ in $\mathbb{R}$, we can find $q \in \mathbb{Q}$ such that $x < q < y$.*

Remark 0.1. So if we choose $y = \sqrt{2}$ and $x_m = \sqrt{2} - \frac{1}{m}$, then we can find $q_m \in \mathbb{Q}$ such that $x_m < q_m < y$. In this way, as $m \to +\infty$, we are approximating $y = \sqrt{2}$ using rational number, this is roughly the approximation we did above.

Proof. If $0 < x < y$. Then the Archimedean properties of $\mathbb{N}$ implies that we can find $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y - x$ or equivalently $ny - nx > 1$.

We claim that there is $m \in \mathbb{N}$ such that $ny > m > nx$. Let $S = \mathbb{N} \cap (nx, +\infty)$. By well-ordering properties, there is $m = \min S \in \mathbb{N}$. Clearly, $m > nx$. If $m \geq ny$, then $m - 1 \geq ny - 1 > nx$. In other word, $m - 1 \in S$ which is impossible as $m = \min S$. Hence,

$$nx < m < ny.$$

So that $q = \frac{m}{n}$ is strictly in between x and y.

If instead $x < 0 < y$, then $q = 0$ is our desired rational number.

If $x < y < 0$, then $-x > -y > 0$. By the first case, there is $-q \in \mathbb{Q}$ such that $-x > -q > -y$ and equivalently, $x < q < y$ for some $q \in \mathbb{Q}$. We are done.

$\square$